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STONE–WEIERSTRASS THEOREMS FOR GROUP-VALUED FUNCTIONS

BY

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ABSTRACT

Constructive groups were introduced by Sternfeld in [6] as a class of metrizable groups G for which a suitable version of the Stone–Weierstrass theorem on the group of G-valued functions $C(X, G)$ remains valid. As a way of exploring the existence of such Stone–Weierstrass-type theorems in this context we address the question raised in [6] as to which groups are constructive and prove that a locally compact group with more than two elements is constructive if and only if it is either totally disconnected or homeomorphic to some vector group $\mathbb{R}^n$. It may therefore be concluded that the Stone–Weierstrass theorem can be extended to some noncommutative Lie groups — exactly to those not containing any nontrivial compact subgroup.

1. Introduction

The Stone–Weierstrass theorem is a classical theorem present in many of the applications of Functional Analysis. It gives natural sufficient conditions for a set of real-valued continuous functions B on a compact space X to be uniformly dense in the set of all real-valued continuous functions $C(X, \mathbb{R})$ or, in other words, to be sufficiently big to allow approximating an arbitrary function $f \in C(X, \mathbb{R})$ with elements of B.

In his paper [6] Sternfeld dealt with the problem of producing such approximation theorems in spaces of group-valued functions. This amounts to finding

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conditions that should be imposed on a set of continuous G-valued functions (G being a metrizable topological group) B to ensure that every continuous function can be approximated with functions in B. Inspired by the results obtained by Sternfeld and Weit [7] for vector-valued functions, Sternfeld introduced the concept of constructive group as a natural way to formulate Stone–Weierstrass-type theorems for group-valued functions.

Definition 1.1: A metrizable group G is said to be **constructive** if, for any compact Hausdorff space K, the following three conditions imposed on a subgroup B of $C(K,G)$ are sufficient to imply that B is uniformly dense:

(1) B separates points of K,
(2) B contains the constants and
(3) B is closed under composition with continuous functions $f: G \to G$. That is to say, $f \circ b \in B$, for every $b \in B$ and $f \in C(G,G)$ (in still other words, f operates on B).

An additive subgroup of $C(K,\mathbb{R})$ stable under the map $x \mapsto x^2$ is immediately a subalgebra of $C(K,\mathbb{R})$ and therefore the classical Stone–Weierstrass theorem shows that $\mathbb{R}$ is a constructive group. It is proved in [6] that the two-element group $\mathbb{Z}_2$ and the circle group $\mathbb{T}$ are examples of nonconstructive groups while the integer group $\mathbb{Z}$ is a *nonclassic* constructive group. We prove in Section 2 that a necessary and sufficient condition for a locally compact connected group to be constructive is to be homeomorphic to a vector group $\mathbb{R}^n$. This is done applying basic homotopy theory to simple compact Lie groups and compact connected Abelian groups to prove via the structure theory of locally compact groups that a constructive connected locally cannot have any nontrivial compact subgroup.

Section 3 is devoted to totally disconnected groups and we prove that a metrizable totally disconnected locally compact group with more than two elements is always constructive. The techniques applied in this Section are more related to set theoretic topology and the key tool is the flexibility to construct continuous functions that appears on zero-dimensional spaces.

The observation made in Section 4 to the effect that constructive groups are either connected or totally disconnected combined with the results in Sections 2 and 3 yields the general result mentioned in the Abstract: the class locally compact constructive groups is composed exactly by totally disconnected groups and groups homeomorphic to $\mathbb{R}^n$.

All groups in this paper will be assumed to be, unless otherwise stated, metrizable and all topological spaces will be Hausdorff. Our terminology and notation are standard. For instance, the symbols $\cdot$ and $^{-1}$ refer, respectively, to the product and inverse operations in a group G and the symbol e will stand for the identity element of G. For general background and concepts not defined here we refer the reader to [4] (for terms related to group theory) and [1] (for terms related to algebraic topology).

2. Connected groups

In studying constructivity for connected locally compact groups we shall first focus on one of their main building blocks, compact Lie groups. The following lemma will simplify the arguments used to prove that compact Lie groups are not constructive.

LEMMA 2.1: *Let G be a topological group admitting a contractible neighbourhood of the identity and let K be a topological space. If $\alpha_n\colon K \to G$ is a uniformly convergent sequence of functions then there is n_0 such that α_n is homotopic to $\alpha = \lim \alpha_n$, for all $n \geq n_0$.*

Proof: Let V be a contractible neighbourhood of e, the identity element of G. Then,

$$N(V) = \{\beta\colon K \to G\colon \beta(k) \cdot \alpha(k)^{-1} \in V \text{ for all } k \in K\}$$

is a neighbourhood of α in the uniform topology and the sequence α_n is eventually contained in $N(V)$. Since V is contractible all the maps $\alpha_n \cdot \alpha^{-1}\colon K \to V$ (for $n \geq n_0$) are homotopic to the constant map e. Consequently, α_n is homotopic to α for $n \geq n_0$. ∎

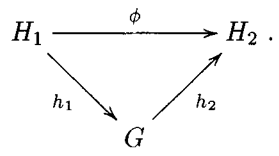
In our next lemma homotopy groups are considered. Recall that the n-th homotopy group $\pi_n(X)$ of a topological space is defined by introducing a suitable group operation in the set of equivalence classes of maps of $I^n \to X$ of a path connected, locally path connected topological space X. We also recall that a continuous map $\phi\colon X \to Y$ between two such topological spaces defines a homomorphism $\phi_*\colon \pi_n(X) \to \pi_n(Y)$ via composition (observe that $\phi \circ f$ is nullhomotopic, i.e. homotopic to a constant map, whenever f is). It is an easy and well-known fact that we shall need later that ϕ_* is an isomorphism of $\pi_n(X)$ into $\pi_n(Y)$ if ϕ is a covering map (even an isomorphism **onto** when $n > 1$). For all these elementary basic facts see [1] or any other manual of algebraic topology.

LEMMA 2.2: *Let G be a topological group. Assume that there are two topological groups H_1 and H_2 such that the following conditions hold:*

(i) *H_1 is locally Euclidean.*

(ii) *Some of the homotopy groups $\pi_n(H_1)$ is nontrivial.*

(iii) *There is a covering map ϕ: $H_1 \to H_2$ that factorizes through G, that is, there are two continuous mappings h_1 and h_2 such that the diagram*



commutes.

Under these hypothesis G is not constructive.

Proof: It will suffice to find a compact space K and a subgroup $B \subseteq C(K, G)$ which is not uniformly dense but satisfies properties (1), (2) and (3) of Definition 1.1. The n-th cube I^n (n chosen such that $\pi_n(H_1)$ is nontrivial) will play the rôle of K.

We define next the subset of $C(I^n, G)$,

$$B = \{\alpha \in C(I^n, G) \colon \alpha \sim 0\},$$

where $\alpha \sim 0$ means that α is nullhomotopic. Note that, since G is not assumed to be path connected, the functions in B are not necessarily pairwise homotopic.

It is clear that B is a subgroup of $C(I^n, G)$ containing the constants that is closed under composition with functions $f \in C(G, G)$. We shall next prove that B separates the points of G but is not dense in $C(I^n, G)$.

To show that B separates points choose two different points in I^n, s_1 and s_2. The group H_1 is assumed to be locally Euclidean and is locally isomorphic to H_2; the latter will consequently have a neighbourhood V of the identity homeomorphic to $\mathbb{R}^k$ for some k. It is easy to find in this situation a map α: $I^n \to V \subseteq H_2$ such that $\alpha(s_1)$ and $\alpha(s_2)$ are different. Since ϕ is a covering mapping, we have that there is a (unique) lifting $\widetilde{\alpha}$: $I^n \to H_1$ of α. The map $\widetilde{\alpha}$ will be nullhomotopic and so will be the map $h_1 \circ \widetilde{\alpha}$. Thus $h_1 \circ \widetilde{\alpha} \in B$ and separates s_1 and s_2.

We finally check that B is not dense in $C(I^n, G)$. Since $\pi_n(H_1)$ is nontrivial we can find α: $I^n \to H_1$ which is not nullhomotopic. The map $h_1 \circ \alpha$ will be then an element of $C(I^n, G)$ that is not in the uniform closure of B. Otherwise, $h_2 \circ h_1 \circ \alpha = \phi \circ \alpha$ would be the uniform limit of a sequence $h_2 \circ \alpha_m$ with $\alpha_m \in B$. But all the maps $h_2 \circ \alpha_m$ are nullhomotopic and, by Lemma 2.1, so should be their uniform limit $\phi \circ \alpha$. Since ϕ is a covering map, the *composition* homomorphism ϕ_* induced by ϕ is an isomorphism of $\pi_n(H_1)$ into $\pi_n(H_2)$ and thus $\phi \circ \alpha \sim 0$ would imply that $\alpha \sim 0$ and that goes against our election of α. ∎

Remark 1: It is an immediate consequence of the above lemma that a constructive Lie group must be homeomorphic to $\mathbb{R}^n$. In particular it may not contain any nontrivial compact subgroup. We shall shortly see that this is a particular case of a more general statement: no constructive locally compact connected group can contain a nontrivial compact subgroup and must, via the structure of locally compact groups, be homeomorphic to $\mathbb{R}^n$.

To extend further the consequences of Lemma 2.2, we need some classical structure theorems (or some consequences of them) for locally compact groups. The first statement of Lemma 2.3 below is just Iwasawa's theorem [5] and the second corresponds to the Borel–Scheerer–Hofmann splitting theorem as found in [4, Theorem 9.39]; see also [3]. Lemma 2.4 is a consequence of Theorem 9.19 of [4].

LEMMA 2.3 (Topological structure of locally compact connected groups): *The following assertions describe the topological structure of locally compact connected groups.*

1. *Let G be a a locally compact connected group and K be a maximal compact subgroup of G. The group G has then a family $R_1, \dots, R_n$ of subgroups topologically isomorphic to $\mathbb{R}$ such that the multiplication map is a homeomorphism of $R_1 \times R_2 \times \cdots \times R_n \times K$ onto G.*
2. *If K is a compact connected group, then K is homeomorphic to the direct product $K' \times A$ where K' is the commutator group of K and A is a compact Abelian group (isomorphic to K/K').*

LEMMA 2.4 (Structure theorem for non-Abelian compact connected groups): *The commutator G' of a compact connected group G is closed and there is a simple simply connected compact Lie group S such that the quotient map $p\colon S \to S/Z(S)$ factorizes through G', that is there are two continuous homomorphisms h_1 and h_2 such that*

$$S \xrightarrow{h_1} G' \xrightarrow{h_2} S/Z(S)$$

with $p = h_2 \circ h_1$.

Let us agree to say that a locally compact group is torus free if it contains no subgroup topologically isomorphic to $\mathbb{T}$, the usual circle group.

LEMMA 2.5: *The maximal compact subgroup of a locally compact connected constructive group must be Abelian and torus free.*

Proof: Let G be a constructive group and K be a maximal compact subgroup. We shall first prove that K must be Abelian and then that it may not contain any subgroup topologically isomorphic to $\mathbb{T}$.

FACT I: *The maximal compact subgroup K must be Abelian.* We employ here the structure theory of locally compact connected groups as described in Lemmas 2.3 and 2.4. If K is not Abelian its commutator subgroup K' will be nontrivial and we may choose (using Lemma 2.4) a simple simply connected compact Lie group S such that the quotient map p: $S \to S/Z(S)$ factorizes through K'. Note that the simplicity of S implies that p is a covering homomorphism (the center must be discrete, hence finite). By Lemma 2.3, G is homeomorphic to $\mathbb{R}^n \times K \times K/K'$ and we can indeed assume that p factorizes through G. Let therefore h_1 and h_2 be two continuous functions

$$S \xrightarrow{h_1} G \xrightarrow{h_2} S/Z(S)$$

such that $p = h_2 \circ h_1$. Compact Lie groups always have some nontrivial homotopy group and Lemma 2.2 applies to show that G is not constructive.

FACT II: *The maximal compact subgroup K must be torus free.* Assume now that K is Abelian and contains a copy of $\mathbb{T}$. Let j denote the topological isomorphism of $\mathbb{T}$ onto a subgroup of K. Every continuous character of $j(\mathbb{T})$ extends to a continuous character of K (this is a standard feature of the duality theory of locally compact Abelian groups; see, e.g., [2, Corollary 24.12]). We may therefore find a continuous character χ: $K \to \mathbb{T}$ such that $\chi \circ j = id_{\mathbb{T}}$. By Lemma 2.3, the character χ can be extended with continuity to the whole group G (although it will no longer be a homomorphism). We have now that the identity mapping id: $\mathbb{T} \to \mathbb{T}$ factorizes through G

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\quad id \quad} & \mathbb{T}\,. \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle \chi} & \\ & G & \end{array}$$

Lemma 2.2 applies again and the proof is done. ∎

Remark 2: Non-Abelian compact connected groups always contain copies of $\mathbb{T}$; the statement of Lemma 2.5 could thus have been summarized by saying that a constructive locally compact connected group must be torus free.

The following structure theorem will be an essential tool for working with torus free groups. A proof is available in [4, Theorems 8.20, 8.22 and 8.37].

LEMMA 2.6 (Structure theorem for compact Abelian groups): *Let G be a compact Abelian group. It is always possible to find a compact totally disconnected group Δ, a topological vector space E and a continuous, open and surjective homomorphism ϕ: $E \times \Delta \to G$ with totally disconnected kernel.*

Moreover, the equality $\ker \phi \cap (E \times \{0\}) = \{0\}$ *holds if G is torus free.*

If in addition G is n-dimensional, E and ϕ can be chosen so that $E = \mathbb{R}^n$ and ϕ is a covering homomorphism.

The term n-dimensional used above refers to the topological dimension of the group. The literature contains an assorted number of (different) definitions of topological dimension referring to different concepts which, however, agree in the case of compact spaces. The situation for compact Abelian groups may be better understood if we recall the fact that the topological dimension of a compact Abelian group is exactly the torsion-free rank of its character group.

To apply Lemma 2.6 it will be necessary to lift certain maps. In those situations we shall recourse to the following version of the monodromy lemma.

LEMMA 2.7 (Theorem 6.1 of [1]): *Let X be a connected, locally path connected topological space and let ϕ: $Y \to Z$ be a covering map between the topological spaces Y and Z. If $\pi_1(X) = \{0\}$, then every continuous map α: $X \to Z$ admits a continuous lift $\tilde{\alpha}$ making the following diagram commutative*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Z\,. \\ {\scriptstyle\tilde{\alpha}}\big\downarrow & \nearrow_{\phi} & \\ Y & & \end{array}$$

THEOREM 2.8: *A connected locally compact group containing a nontrivial compact subgroup is not constructive.*

Proof: Let G be a connected locally compact group and let K be a (nontrivial) maximal compact subgroup of G. After Lemma 2.5 we may assume that K is Abelian and torus free.

We observe first that K must have an n-dimensional quotient K_1 that is not path connected. To see this, it suffices to find a finitely generated subgroup of $\widehat{K}$ (the character group of K) that is not free because every path connected compact metric group is a torus (see, e.g., [4, Theorem 8.46]) and character groups of free groups are tori (accounts of Pontryagin duality on which these facts are based can be found, for instance, in [2, Chapter 6] or [4, Chapter 7]). Suppose no such subgroup can be found in $\widehat{K}$. Groups with this property are called $\aleph_1$-free (see Definition A1.63 and Proposition A1.64 of [4]) and $\aleph_1$-free groups have the property that homomorphisms into $\mathbb{Z}$ separate the points [4, Proposition A1.66]. It is obvious then that the character group of a torus free group cannot be $\aleph_1$-free and that $\widehat{K}$ must have a finitely generated nonfree subgroup.*

We proceed now to show that G is not constructive defining

$$B = \{\alpha\colon K \to G\colon \alpha(K) \text{ is locally path connected}\}.$$

It is obvious that B is a subgroup that contains the constants and, since any mapping between compact groups is a quotient map, it is also clear that $C(G,G)$ operates on B.

We now prove that B separates points but is not dense in $C(K,G)$.

To check that B separates points, we apply the structure Lemma 2.6 to K and find a continuous, open and surjective homomorphism $\phi\colon E \times \Delta \to K$ such that $\ker\phi \cap (E \times \{0\}) = \{0\}$ with E some topological vector space and Δ some compact totally disconnected group. If x and y are two different elements of K we take a continuous real valued function σ with $\sigma(x) = 0$ and $\sigma(y) \neq 0$. Denoting by $j\colon \mathbb{R} \to E \times \{0\}$ a natural embedding of $\mathbb{R}$ in $E \times \{0\}$ it is clear that $\alpha\colon = \phi \circ j \circ \sigma$ is in B and separates x and y.

To check that B is not dense in $C(K,G)$, we shall prove that the inclusion mapping $i\colon K \to G$ cannot be approximated by functions in B. We apply again the structure Lemma 2.6, this time to the n-dimensional group K_1. This produces a totally disconnected compact group Δ_1 and a covering homomorphism $\phi_1 : \mathbb{R}^n \times \Delta_1 \to K_1$.

By Iwasawa's theorem (Lemma 2.3) there is some integer $m \geq 0$ such that G is homeomorphic to $\mathbb{R}^m \times K$. We fix that m and trivially extend the covering homomorphism ϕ_1 to a covering homomorphism $\phi_1\colon R^{n+m} \times \Delta_1 \to \mathbb{R}^m \times K_1$. In the same vein we can regard the quotient homomorphism $p\colon K \to K_1$ as a quotient homomorphism $p\colon \mathbb{R}^m \times K \to \mathbb{R}^m \times K_1$. Let now $V = W \times W'$ be a basic neighbourhood of the identity of G (topologically viewed as $\mathbb{R}^m \times K$). Since ϕ_1 is a covering mapping, V can be chosen so that $p(V) = p(W \times W') = \phi_1(V_0)$ where V_0 is a neighbourhood of the identity of $\mathbb{R}^{n+m} \times \Delta$. The identity components of $p(V)$ will be thus contractible.

* The recourse to $\aleph_1$-free groups is unnecessary if we lean on the metrizability of K. By an old theorem of Pontryagin $\aleph_1$-freeness is equivalent to freeness for countable groups; see also [4, Proposition A1.64]. In order to make clear how our techniques apply beyond the metric case, we have chosen this approach independent of the metrizability of K; see Remark 3 at the end of the paper.

If i were in the uniform closure of B, it would be possible to find $\alpha \in B$ such that $\alpha(x) - x \in V$ for every $x \in K$. As a consequence, $p \circ (\alpha - i)$ would define a continuous mapping of K into $p(V)$. Since K is connected, $p \circ (\alpha - i)$ would actually define a map of K into a connected component of $p(V)$ which, by the argument in the previous paragraph, is contractible. We would conclude therefore that we can find a lift $\widetilde{\alpha - i}$ of $p \circ (\alpha - i)$ such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{p\circ(\alpha-i)} & p(V) \\ {\scriptstyle \widetilde{\alpha-i}}\downarrow & \nearrow_{\phi_1} & \\ V_0 & & \end{array}$$

commutes.

On the other hand, $(p\circ\alpha)(K)$ is a locally path connected metric space and the monodromy lemma (Lemma 2.7, applied to the inclusion mapping of $(p \circ \alpha)(K)$ into $\mathbb{R}^m \times K_1$) implies that $p \circ \alpha$ admits a lift $\widetilde{\alpha}$ making the diagram

$$\begin{array}{ccc} K & \xrightarrow{p\circ\alpha} & \mathbb{R}^m \times K_1 \\ {\scriptstyle \widetilde{\alpha}}\downarrow & \nearrow_{\phi_1} & \\ \mathbb{R}^{n+m} \times \{e\} & & \end{array}$$

commutative.

We observe finally that $\phi_1 \circ (\widetilde{\alpha} - \widetilde{\alpha - i}) = p$ which means that $p(K)$ should be contained in the path component of K_1 (note that $\phi_1(\mathbb{R}^{n+m} \times \{e\})$ is always contained in the path component of $\mathbb{R}^m \times K_1$), but this is plainly impossible since p is a quotient map and K_1 is not path connected.

We conclude thus that the inclusion mapping is not in the uniform closure of B. ∎

THEOREM 2.9: *A connected locally compact group is constructive if and only if it is homeomorphic to* $\mathbb{R}^n$, *for some* n.

Proof: Theorem 2.8 proves that a constructive connected locally compact group has a trivial maximal compact subgroup, thus Iwasawa's theorem (see Lemma 2.3) implies that it must be homeomorphic to $\mathbb{R}^n$.

To prove the other direction consider a locally compact group homeomorphic to $\mathbb{R}^n$. Again by statement (1) of Lemma 2.3 there will be some subgroups $R_1, \ldots, R_n$ of G such that the multiplication map $(x_1, \ldots, x_n) \mapsto x_1 \cdots x_n$ is a homeomorphism onto G.

Let K be any compact topological space and B a subgroup of $C(K,G)$ that satisfies (1), (2) and (3) of Definition 1.1. Now, let B_j denote the subset of B whose elements are maps with range contained in R_j. The sets B_j can be regarded as subgroups of $C(K,R_j)$ that contain the constants since B did so.

If x and y are different points of K there is some $\alpha \in B$ with $\alpha(x) \neq \alpha(y)$. Now choose a continuous map ϕ of G into R_j that separates $\alpha(x)$ and $\alpha(y)$ (this amounts to separating two points of $\mathbb{R}^n$ with a real-valued continuous mapping). Since $C(G,G)$ operates on B, $\phi \circ \alpha \in B_j$ and separates x and y. Thus B_j is a point-separating subgroup of $C(K,R_j)$.

Let $h\colon R_j \to R_j$ be any continuous map and let $\alpha \in B_j$. Since G is homeomorphic to $R_1 \times \cdots \times R_n$ we can extend trivially h to a map $\bar{h}$ of G into R_j. Now $C(G,G)$ operates on B and α also belongs to B; it is thus clear that $\bar{h} \circ \alpha \in B_j$, but $\bar{h} \circ \alpha = h \circ \alpha$ and we deduce that $C(R_j,R_j)$ operates on B_j.

Having proved that B_j is a subgroup of $C(K,R_j)$ with properties (1), (2) and (3) of Definition 1.1 we may invoke the classical Stone–Weierstsrass theorem to conclude that each B_j is uniformly dense in $C(K,R_j)$. Now choose some $f \in C(K,G)$. It is now easy (just compose with the inverse of the multiplication map and with the corresponding projection) to define for each j a map $f_j \in C(K,H_j)$, $1 \leq j \leq n$, such that $f(x) = f_1(x) \cdot f_2(x) \cdots f_n(x)$; each f_j will be the uniform limit of a sequence of maps belonging to B_j (and thus also to B) and therefore $f = f_1 \cdot f_2 \cdots f_n$ will be in the uniform closure of B. ∎

3. Totally disconnected groups

Our aim in this section is to show in a dramatic way how the nonconstructive group $\mathbb{Z}_2$ is an isolated case. This will be indeed the only example among locally compact totally disconnected groups. The statement of the following lemma points to the basic idea after the proof of that fact and its proof, reminiscent of Proposition 1 of [6], takes care of the necessary technicalities.

LEMMA 3.1: *Let G be a zero-dimensional topological group with $|G| > 2$ and let K be a compact Hausdorff space. If B is a subgroup of $C(K,G)$ that separates the points of K and such that $C(G,G)$ operates on B and K_0 is a clopen subset of K, then for every $e \neq a \in G$, there is a function $F_a \in B$ such that*

$$F_a(K_0) = \{a\} \quad \textit{and} \quad F_a(K \setminus K_0) = \{e\}.$$

Proof: If x_1 and x_2 are two different elements of a zero-dimensional topological space X and y_1, y_2 are arbitrary elements belonging to another topological space

Y, it is always possible to find a continuous function $f \in C(X,Y)$ with $f(x_i) = y_i$, $i = 1,2$. This fact will be used freely throughout this proof with $X = Y = G$.

First of all choose $b \in G$, such that $b \neq a$ and $b \neq a^2$.

The elements of B separate points of K, and it is easy to find for every $x \in K_0$ and every $y \notin K_0$ a function $v_{x,y} \in B$ with $v_{x,y}(x) = a$ and $v_{x,y}(y) = b$. A simple compactness argument produces a covering by clopen sets

$$K_0 \subseteq \bigcup_{j=1}^{n} A_{j,y}$$

and a sequence of continuous functions $f_{j,y}: K \to G$ with the property

$$f_{j,y}(A_{j,y}) = \{a\} \quad \text{and} \quad f_{j,y}(y) = b.$$

CLAIM: *For every $y \notin K_0$, it is possible to find $H_y \in B$ such that $H_y(K) \subseteq \{a, a^2, ab, b\}$, $H_y(K_0) \subseteq \{a, a^2, ab\}$ and $H_y(y) = b$.*

To prove the claim we shall define recursively a sequence of functions $H_{j,y}$ with the property

$$H_{j,y}(y) = b \quad \text{and} \quad H_{j,y}(A_{m,y}) \subseteq \{a, a^2, ab\} \quad \text{if } m \leq j.$$

Once this sequence is defined the function $H_{n,y}$ will be the desired function H_y.

To start with, consider two continuous functions $j_1: G \to \{a,b\}$ and $j_2: G \to \{a,e\}$ satisfying the relations

$$j_i(a) = j_i(a^2) = j_i(ab) = a, \quad i = 1,2, \quad j_1(b) = b, \quad \text{and} \quad j_2(bN_0) = e$$

where N_0 is some clopen neighbourhood of the identity not containing $b^{-1}a$, $b^{-1}ab$ or $b^{-1}a^2$ (so that $\{a, ab, a^2\} \cap bN_0 = \emptyset$).

Next, we take $f_{1,y}$ as $H_{1,y}$ and define $H_{m,y} = (j_2 \circ f_{m,y}) \cdot (j_1 \circ H_{m-1,y})$. Certainly, $H_{m,y} \in B$ since B is a subgroup and functions in $C(G,G)$ operate on B. If $k \in A_{j,y}$ with $j < m$, then $H_{m,y}(k) \in \{a, ab\}$ because $H_{m-1,y}(k) \in \{a, a^2, ab\}$ by recursive hypothesis and $j_1(a) = j_1(ab) = j_1(a^2) = \{a\}$ while the only possible values for $j_2(f_{2,y}(k))$ are a and e. If $k \in A_{m,y}$, the same conclusion is obtained, for $j_2(f_{m,y}(k)) = a$. Finally $H_{m,y}(y) = j_2(b) \cdot j_1(b) = b$.

Once the claim is proved we use an analogous construction to define the function claimed in the conclusion of this lemma. Choose another clopen neighbourhood N_1 of the identity in G with $N_1 \subseteq N_0$ and such that $a^2N_1 \cap \{a, b, ab, ba\} = \emptyset$. For each $y \notin K_0$ denote by Z_y the clopen $Z_y = H_y^{-1}(bN_1)$; it will be possible, by compactness, to find $y_1, y_2, \ldots, y_m$ such that

$$K \setminus K_0 \subseteq \bigcup_{j=1}^{m} Z_{y_j}.$$

Consider now a function $j_3 \in C(G,G)$ with $j_3(a^2) = a$ and $j_3(x) = b$ if $x \notin a^2 N_1$. With these ingredients we define inductively

$$F_1 = H_{y_1} \quad \text{and} \quad F_j = j_3 \circ (j_1 \circ H_{y_j} \cdot j_2 \circ F_{j-1}).$$

If $x \in Z_{y_1}$, $F_1(x) = b$ (note that $F_1(x) \in bN_1 \cap H_{y_1}(K) = \{b\}$), while $F_1(x) = a$ if $a \in K_0$. Assume as *inductive hypothesis* that $F_{j-1}(x) = b$ if $x \in Z_{y_r}$ with $r \leq j-1$ and $F_{j-1}(x) = a$ if $x \in K_0$; we shall prove that the same relations hold for F_j.

If $x \in Z_{y_r}$ and $r \leq j-1$ we know from the inductive hypothesis that $F_{j-1}(x) = b$, thus $F_j(x) = j_3(v \cdot b)$, where v is either a or e (the range of j_2) and hence $F_j(x) = b$. If $x \in Z_{y_j}$, then $F_j(x) = j_3(e \cdot w) = b$ because w is in the range of j_1 that consists only of a and b. And for $x \in K_0$ we have $F_j(x) = j_3(a^2) = a$ since $H_{y_j}(a) \in \{a, a^2, ab\}$ and, by definition, $j_2(a) = j_2(ab) = j_2(a^2) = a$.

Having constructed a sequence F_j with $F_j(Z_{y_r}) = \{b\}$ if $r \leq j$ and $F_j(K_0) = \{a\}$, it is clear that the function F_m maps K_0 onto $\{a\}$ and $K \setminus K_0$ onto b; composing the function F_m with some $j \in C(G,G)$ mapping a onto a and b onto e, the lemma follows. ∎

THEOREM 3.2: *A locally compact, totally disconnected metric group G with $|G| > 2$ is constructive.*

Proof: Let K be a compact Hausdorff space and let B be a subgroup of G satisfying properties (3), (1) and (2) of Definition 1.1. We should prove that B is dense in $C(K,G)$.

A locally compact totally disconnected metric group G admits a neighbourhood basis at the identity $\{N_1 \supseteq \cdots \supseteq N_k \supseteq \cdots\}$ consisting of open subgroups [2, Theorem 7.7] and is thus a zero-dimensional topological space (as a matter of fact, every totally disconnected locally compact space is zero-dimensional).

Let f be any function in $C(K,G)$. To see that f may be uniformly approximated from within B, it will suffice to check that for every k, there is some $f_k \in B$ such that, denoting by π_k the projections onto G/N_k (discrete espace of left cosets), $\pi_k \circ f_k = \pi_k \circ f$, for in that case f will be the uniform limit of the sequence $\{f_k\}_{k<\omega}$.

The function $\pi_k \circ f$ maps the compact space K into the discrete space G/N_k. It follows that there are finitely many clopen subsets $K_1, \dots, K_r$ in K such that $K = \bigcup K_i$ and $\pi_k \circ f$ takes the constant value a_i on K_i. By Lemma 3.1 we can find $h_i \in B$ such that $h_i(K_i) = a_i$ and $h_i(K \setminus K_i) = e$. It is then clear that $\pi_k \circ f = \pi_k \circ (h_1 \cdot \ldots \cdot h_r)$ and the theorem is proved. ∎

4. The general case

Sections 2 and 3 deal with connected and totally disconnected groups separately. We are now in position to put these results together by means of the following theorem that, when applied to (locally) compact groups, will imply that constructive groups are either homeomorphic to $\mathbb{R}^n$ or totally disconnected. Let G_0 denote the connected component of the identity of the topological group G.

THEOREM 4.1: *A constructive group is either connected or totally disconnected.*

Proof: Let G be a nonconnected topological group with $|G_0| > 1$. We shall prove that G is not constructive. To this in turn, we consider a discrete space of two elements, say $K = \{1, 2\}$, and the subset B of $C(K, G)$ defined as

$$B = \{f \in C(K,G) \colon f(K) \subseteq x_f \cdot G_0 \text{ for some } x_f \in G\}$$

and we shall show that B is a proper closed subgroup of $C(K,G)(\cong G^2)$ which satisfies properties (3), (1) and (2) of Definition 1.1.

Notice firstly that B is a subgroup simply because G_0 is a normal subgroup of G and that, since G is not connected, it will contain at least two different connected components which implies that $B \neq C(K,G)$. Thus, B is a proper subgroup of $C(K,G)$.

Furthermore, the connected components form a cover of G and $|G_0| > 1$, whence we have that B contains the constants and separates the points of K. To finish the proof, we only need to show therefore that (a) B is closed under composition with functions $f \in C(G,G)$, and (b) B is closed in $C(K,G)$. Actually, (a) is an easy consequence of the fact that continuous images of connected sets are also connected. To see (b), let $\{f_n\}_{n=1}^{\infty}$ be a sequence in B converging to $f \in C(K,G)$. By definition of B, we have $f_n(2) = f_n(1) \cdot y_n$ with $y_n \in G_0$ for $n = 1, 2, \ldots$. Since $\{f_n(i)\}_{n=1}^{\infty}$ converges to $f(i)$ for each $i = 1, 2$ and G_0 is closed, the sequence $\{y_n\}_{n=1}^{\infty}$ converges to a point $y \in G_0$. Then $f(2) = f(1) \cdot y$ and, consequently, $f(K) \subseteq f(1) \cdot G_0$. Thus, $f \in B$. ∎

COROLLARY 4.2: *For a locally compact group G with $|G| > 2$, the following assertions are equivalent:*

(1) *G is constructive.*
(2) *G is either totally disconnected or a Lie group homeomorphic to $\mathbb{R}^n$, for some n.*

Proof: The proof follows from a simple juxtaposition of Theorems 4.1 and 2.8 for one direction and of Theorem 2.9 for the other. ∎

Remark 3: Following the definition of constructive group given by Sternfeld [6], we have chosen the class of metrizable groups as a natural setting to state Stone–Weierstras-type theorems for group-valued functions. Sternfeld's definition nevertheless makes sense for nonmetrizable groups and it seems interesting to point out that our proofs apply with minor changes in this general context, that is, all the results in this paper may be carried out for nonmetrizable groups with little additional effort.

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